ON THE CLASS SEMIGROUP OF ROOT-CLOSED WEAKLY KRULL MORI MONOIDS

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Abstract. A C-monoid is a suitably defined submonoid of a factorial monoid with finite reduced class semigroup. This monoid plays a key role in an arithmetical investigation of a large class of Mori domains. It is well understood that a C-monoid is Krull if and only if the reduced class semigroup coincides with the (v-)class group of a Krull monoid, and the arithmetic of a Krull monoid can be determined by the structure of its (v-)class group. The finiteness of the reduced class semigroup allows us to prove the similar arithmetical finiteness for a general C-monoid to results known in the Krull case. Recently, the algebraic structure of the reduced class semigroup has begun to be studied for a non-Krull C-monoid. Every Krull monoid is a root-closed weakly Krull Mori monoid, and under the mild conditions, a root-closed weakly Krull Mori monoid is a C-monoid. In this paper, we study the structure of a root-closed weakly Krull Mori monoid and of its class semigroup.

1. Introduction

A C-monoid H is a submonoid of a factorial monoid F such that $H^\times = F^\times \cap H$ and the reduced class semigroup is finite. An integral domain is said to be a C-domain if its monoid of non-zero elements is a C-monoid. C-monoids have been introduced in [14, 23] as multiplicative models to study the arithmetic of higher-dimensional non-integrally closed Noetherian domains (or non-completely integrally closed Mori domains). Let R be a Mori domain with $\mathfrak{f} = (R : \widehat{R}) \neq \{0\}$. If both the v-class group $\mathcal{C}_v(\widehat{R})$ and the residue ring $\widehat{R}/\mathfrak{f}$ are finite, then R is a C-domain [15, Theorem 2.11.9], and the converse holds true for non-local semilocal Noetherian domains [29, Corollary 4.5]. The concept of C-domains has been generalized to rings with zero divisors, and we refer the reader to [17] for a detailed study.

Let H be a C-monoid. Then, H is a Mori monoid, and H is completely integrally closed if and only if its reduced class semigroup is a group [15, Section 2.9]. Thus, every Krull monoid with finite (v-)class group is a C-monoid, and the reduced class semigroup coincides with the (v-)class group. Moreover, the arithmetic of such a monoid can be determined to a large extent by the structure of its (v-)class group (see, [30, 20] for a survey). However, for a non-Krull C-monoid, we only have the arithmetical finiteness results which were derived from the finite condition of the reduced class semigroup (see [15, Section 3.3 and 4.6] and [8, 7, 23, 9]).

In recent years, the algebraic structure of the reduced class semigroup of a C-monoid has begun to be studied. The monoid $\mathcal{B}(G)$ of product-one sequences over a finite group G was the first class of C-monoids for which we have an insight into a structural relationship between a C-monoid and its reduced class semigroup. Among others, it was proved that the reduced class semigroup of $\mathcal{B}(G)$ is Clifford, i.e., it is a union of groups, if and only if $\mathcal{B}(G)$ is a seminormal monoid if and only if the commutator subgroup of G has at most two elements [25, Corollary 3.12]. More generally, the first two conditions were successfully generalized to a general C-monoid, i.e., a C-monoid is seminormal if and only if its reduced class semigroup is Clifford [19, Theorem 1.1].

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In this present paper, we study the algebraic structure of the reduced class semigroup of specific C-monoids. Every Krull monoid is a root-closed (and so, seminormal) weakly Krull Mori monoid, and the arithmetic of a seminormal weakly Krull Mori monoid has been studied in [16, 18]. A Weakly Krull domain R possesses a defining system of finite character consisting of localizations of R at minimal primes (see, [22, Chapter 22]). If R is a Mori domain with $(R : \widehat{R}) \neq \{0\}$, then multiplicative models of localizations of R at minimal primes are finitely primary. After putting together the required background in Section 2, we study the root-closure of finitely primary monoids, as the local case of a weakly Krull Mori domain, in Section 3. Among other things, we describe a relationship between the root-closure and the seminormalization of a finitely primary monoid, and we show that a root-closed finitely primary monoid is a C-monoid (see Lemma 3.1). Moreover, we provide the structure of the reduced class semigroup of root-closed finitely primary monoids (see, Theorem 3.4 and Corollary 3.6). In Section 4, we study the global case for root-closed weakly Krull Mori monoids. Among other things, we provide the structural information of the reduced class semigroup of root-closed weakly Krull Mori monoids that are C-monoids (see, Theorem 4.4).

2. Definitions and preliminaries

In this preliminary, we gather the key notions and the required terminology, and our main references are [15, 22]. To begin with, $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For integers $a, b \in \mathbb{Z}$, $[a, b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$ means the discrete interval.

Semigroups and Monoids. Throughout this paper, all semigroups are commutative and they have an identity element. Let $\mathcal{C}$ be a semigroup with identity element 1. Then, $\mathcal{C}^\times$ denotes the group of invertible elements of $\mathcal{C}$, and $\mathcal{C}$ is called *reduced* if $\mathcal{C}^\times = \{1\}$. An element $e \in \mathcal{C}$ is *idempotent* if $e^2 = e$, and we denote by $\mathsf{E}(\mathcal{C})$ the set of all idempotent elements of $\mathcal{C}$. We say that $\mathcal{C}$ is *cancellative* if every element $a \in \mathcal{C}$ is cancellative, i.e., $ab = ac$ for $b, c \in \mathcal{C}$ implies that $b = c$. For a subset $U \subseteq \mathcal{C}$, we denote by $[U]$ the smallest subsemigroup of $\mathcal{C}$ containing U, i.e., $[U]$ consists of all products $u_1 \cdots u_n$, where $n \in \mathbb{N}_0$ and $u_1, \ldots, u_n \in U$. The semigroup $\mathcal{C}$ is said to be *finitely generated* if $\mathcal{C} = [U]$ for a finite subset $U \subseteq \mathcal{C}$.

A *monoid* means a cancellative semigroup. Let H be a monoid. Then, $\mathsf{q}(H)$ denotes the quotient group of H, and $H_{\mathrm{red}} = H/H^\times = \{aH^\times \mid a \in H\}$ denotes the associated reduced monoid of H. If H is reduced, then we set $H = H_{\mathrm{red}}$. We denote by

- $H' = \{x \in \mathsf{q}(H) \mid \text{there exists an integer } N \in \mathbb{N} \text{ such that } x^n \in H \text{ for all } n \ge N\}$ the *seminormalization* of H, by
- $\widetilde{H} = \{x \in \mathsf{q}(H) \mid x^N \in H \text{ for some } N \in \mathbb{N}\}$ the *root closure* of H, and by
- $\widehat{H} = \{x \in \mathsf{q}(H) \mid \text{there exists an element } c \in H \text{ such that } cx^n \in H \text{ for all } n \in \mathbb{N}\}$ the *complete integral closure* of H.

Clearly, $H \subseteq H' \subseteq \widetilde{H} \subseteq \widehat{H} \subseteq \mathsf{q}(H)$, and the monoid H is said to be *seminormal* (resp., *root-closed*, and *completely integrally closed*) if $H = H'$ (resp., $H = \widehat{H}$, and $H = \widehat{H}$).

For subsets $X, Y \subseteq \mathsf{q}(H)$, we set $(X : Y) = \{x \in \mathsf{q}(H) \mid xY \subseteq X\}$, $X^{-1} = (H : X)$, and $X_v = (X^{-1})^{-1}$. A subset $X \subseteq \mathsf{q}(H)$ is said to be

- *H-fractional* if there exists an element $c \in H$ such that $cX \subseteq H$,
- a *fractional v-ideal* of H if X is H-fractional and $X_v = X$, and
- a *v-ideal* of H if $X \subseteq H$ and $X_v = X$.

We denote by $\mathcal{F}_v(H)$ the semigroup of fractional v-ideals of H with v-multiplication, i.e., $X \cdot_v Y = (XY)_v$ for any $X, Y \in \mathcal{F}_v(H)$, and by $\mathcal{I}_v(H)$ the subsemigroup of v-ideals of H. Then, $\mathcal{I}_v^*(H) = \mathcal{I}_v(H) \cap \mathcal{F}_v(H)^\times$ is the monoid of v-invertible v-ideals of H, and $\mathcal{F}_v(H)^\times = \mathsf{q}\big(\mathcal{I}_v^*(H)\big)$. Above constructions can be generalized to monoids of r-ideals for a general ideal system r, and we refer the reader to [28, 12] for a recent progress on the algebraic and arithmetic properties of monoids of ideals.

The monoid H is said to be a

- *Mori monoid* if it satisfies the Ascending Chain Condition (ACC) on v-ideals, and
- *Krull monoid* if it is a completely integrally closed Mori monoid.

For a set P, we denote by $\mathcal{F}(P)$ the free abelian monoid with basis P. If $P = \{p_1, \ldots, p_\ell\}$ is finite, then we set $\mathcal{F}(P) = \mathcal{F}\big(\{p_1, \ldots, p_\ell\}\big) = [p_1, \ldots, p_\ell]$. A monoid F is factorial if and only if F_{red} is free abelian. Let $F = F^\times \times \mathcal{F}(P)$ be a factorial monoid. Then, every element $a \in F$ has a unique representation of the form

$$a = \varepsilon \prod_{p \in P} p^{\mathsf{v}_p(a)} \quad \text{with} \quad \varepsilon \in F^\times \ \text{ and } \ \mathsf{v}_p(a) = 0 \quad \text{for almost all} \quad p \in P\,.$$

A monoid homomorphism $\varphi \colon H \to D$ is said to be

- a *divisor homomorphism* if $a, b \in H$ and $\varphi(a) \mid \varphi(b)$ in D implies that $a \mid b$ in H.
- *cofinal* if, for every $x \in D$, there exists $a \in H$ such that $x \mid \varphi(a)$ in D.
- a *divisor theory* if D is free abelian, φ is a divisor homomorphism, and for all $\alpha \in D$, there are $a_1, \ldots, a_m \in H$ such that $\alpha = \gcd\big(\varphi(a_1), \ldots, \varphi(a_m)\big)$.

Let $H \subseteq D$ be monoids. Then, H is said to be *saturated* (resp., *cofinal*) if the inclusion $H \hookrightarrow D$ is a divisor homomorphism (resp., cofinal). It is easy to see that $H \subseteq D$ is saturated if and only if $H = \mathsf{q}(H) \cap D$.

Class groups. Let $H \subseteq D$ be monoids. Then, the group $\mathsf{q}(D)/\mathsf{q}(H) = \{x\mathsf{q}(H) \mid x \in \mathsf{q}(D)\}$ is called the *class group* of H in D, and we usually write this group additively. We define

$$D/H = \{a\mathsf{q}(H) \mid a \in D\} \subseteq \mathsf{q}(D)/\mathsf{q}(H)\,,$$

and then it is easy to show that $H \subseteq D$ is cofinal if and only if D/H is a group. In particular, if D/H is finite, or if $\mathsf{q}(D)/\mathsf{q}(H)$ is torsion, then $D/H = \mathsf{q}(D)/\mathsf{q}(H)$ ([15, Corollary 2.4.3]). If $\mathcal{H}(H) = \{aH \mid a \in H\}$ is the monoid of principal ideals of H, then $\mathcal{H}(H) \subseteq \mathcal{I}_v^*(H)$ is a saturated and cofinal submonoid, and we have that

$$\mathcal{C}_v(H) = \mathcal{F}_v(H)^\times/\mathsf{q}\big(\mathcal{H}(H)\big) = \mathsf{q}\big(\mathcal{I}_v^*(H)\big)/\mathsf{q}\big(\mathcal{H}(H)\big) = \mathcal{I}_v^*(H)/\mathcal{H}(H)\,,$$

which is called the *v-class group* of H.

It is well known that a monoid H is Krull if and only if H has a divisor theory ([15, Theorem 2.4.8]). Suppose that H is a Krull monoid. Then, there exists a free abelian monoid $\mathcal{F}(P)$ such that the inclusion $H_{\mathrm{red}} \hookrightarrow \mathcal{F}(P)$ is a divisor theory. In this case, $\mathcal{F}(P)$ is uniquely determined up to isomorphism, and the class group $\mathsf{q}\big(\mathcal{F}(P)\big)/\mathsf{q}\big(H_{\mathrm{red}}\big)$ of H is isomorphic to the v-class group $\mathcal{C}_v(H)$ of H (see, [15, Section 2.4]). It is well known that a Krull monoid H is factorial if and only if the v-class group $\mathcal{C}_v(H)$ is trivial. Thus, the class group measures how far away H is from being factorial, and so it plays a crucial role in the study of the arithmetic of Krull monoids and domains. If every class of the class group of H contains a prime divisor, then the combinatorial object, named the monoid of zero-sum (or product-one) sequences over the class group (see, before Remark 3.3 for the short introduction), reflects the arithmetic of H via transfer homomorphism [15, Theorem 3.4.10]. We refer the reader to [11, 20] for a survey on the arithmetic of Krull monoids and to [5] for a recent progress on prime divisors of Krull monoid algebras.

Class semigroups and C-monoids. A detailed presentation can be found in [15, Sections 2.8 and 2.9]. Let F be a factorial monoid, and $H \subseteq F$ be a submonoid. Any two elements $y, y' \in F$ are called H-equivalent, denoted by $y \sim_H y'$, if $y^{-1}H \cap F = (y')^{-1}H \cap F$, i.e., for every $x \in F$, we have that $xy \in H$ if and only if $xy' \in H$. Then H-equivalence is a congruence relation on F. For $y \in F$, let $[y]_H^F$ denote the congruence class of y, and let

$$\mathcal{C}(H,F) = \{[y]_H^F \mid y \in F\} \qquad \text{and} \qquad \mathcal{C}^*(H,F) = \big\{[y]_H^F \mid y \in (F \setminus F^\times) \cup \{1\}\big\}\,.$$

Then, $\mathcal{C}(H,F)$ is a commutative semigroup with identity element $[1]_H^F$, called the *class semigroup* of H in F, and $\mathcal{C}^*(H,F) \subseteq \mathcal{C}(H,F)$ is a subsemigroup, called the *reduced class semigroup* of H in F. As usual, the (reduced) class semigroups are written additively.

A monoid H is said to be a *C-monoid* defined in F if it is a submonoid of a factorial monoid $F = F^\times \times \mathcal{F}(P)$ such that $H^\times = F^\times \cap H$ and $\mathcal{C}^*(H,F)$ is finite. If H is a C-monoid defined in F and $\mathcal{C}^*(H,F)$ is a group, then H is a Krull monoid, and conversely, every Krull monoid with finite (v-)class group is a C-monoid (in this case, the (v-)class group and the class semigroup coincide) (see, [15, Theorem 2.9.12]). Let H be a C-monoid defined in F. Then, there exist $\alpha \in \mathbb{N}$ and a subgroup $V \subseteq F^\times$ such that

$$H^\times \subseteq V\,, \quad (F^\times : V) \mid \alpha\,, \quad V(H \setminus H^\times) \subseteq H\,, \quad \text{and} \tag{2.1}$$

$$q^{2\alpha}F \cap H = q^\alpha(q^\alpha F \cap H) \ \text{ for all } \ q \in F \setminus F^\times\,. \tag{2.2}$$

(see, [15, Proposition 2.8.11]). In particular, if $p \in P$ and $a \in p^\alpha F$, then $a \in H$ if and only if $p^\alpha a \in H$. We say that H is *dense* in F (this is a minimality condition on F, see [15, Theorem 2.9.11]) if $\mathsf{v}_p(H) \subseteq \mathbb{N}_0$ is a numerical monoid for every $p \in P$, i.e., $\mathsf{v}_p(H) \subseteq \mathbb{N}_0$ is an additive submonoid such that $\mathbb{N}_0 \setminus \mathsf{v}_p(H)$ is finite for every $p \in P$.

The following lemma describes the algebraic properties of C-monoids, and its proof can be found in [15, Theorems 2.9.11 and 2.9.13].

Lemma 2.1. *Let H be a dense C-monoid defined in a factorial monoid $F = F^\times \times \mathcal{F}(P)$.*

1. *H is a Mori monoid with $(H : \widehat{H}) \neq \emptyset$.*
2. *$\widehat{H} = \mathsf{q}(H) \cap F$ is a Krull monoid with finite v-class group $\mathcal{C}_v(\widehat{H})$.*
3. *The map $\partial \colon \widehat{H} \to \mathcal{F}(P)$, defined by*

$$\partial(a) \;=\; \prod_{p \in P} p^{\mathsf{v}_p(a)}\,,$$

is a divisor theory, and there exists an epimorphism $\mathcal{C}^*(H,F) \to \mathcal{C}_v(\widehat{H})$.

In particular, F_{red}, and so $\mathcal{C}^*(H,F)$, is uniquely determined by H up to isomorphism.

Let R be an integral domain. Then, the set of all non-zero elements $R^\bullet$ of R is a multiplicative monoid, and an ideal-theoretic relationship between R and $R^\bullet$ has received wide attention in the literature (see [15, 22] for the monographs). The domain R is said to be a

- *Krull domain* if $R^\bullet$ is a Krull monoid,
- *C-domain* if $R^\bullet$ is a C-monoid.

If R is a non-local semilocal Noetherian domain, then R is a C-domain if and only if the (v-)class group of $\widehat{R}$ and the residue ring $R/(R : \widehat{R})$ are both finite ([29, Corollary 4.5]). More generally, C-rings of commutative rings with zero divisors can be defined in the same manner as the domain case, and we refer the reader to [17] for a detailed study.

3. The root-closed finitely primary monoids

The monoid H is said to be *finitely primary of rank s and exponent α* if there exist $s, \alpha \in \mathbb{N}$ such that H is a submonoid of a factorial monoid $F = F^\times \times [p_1, \ldots, p_s]$ with s pairwise non-associated prime elements $p_1, \ldots, p_s$ satisfying

$$H \setminus H^\times \subseteq (p_1 \cdots p_s)F \qquad \text{and} \qquad (p_1 \cdots p_s)^\alpha F \subseteq H\,.$$

If H is finitely primary of rank s, then obviously $F = \widehat{H}$ and $s = \big|\mathfrak{X}(\widehat{H})\big|$, where $\mathfrak{X}(\widehat{H})$ is the set of non-empty minimal prime ideals of $\widehat{H}$. Finitely primary monoids are multiplicative models of one-dimensional local domains (see Lemma 4.5), and they play a key role in the study of the structure of the monoid of v-invertible v-ideals of weakly Krull Mori domains (see Corollary 4.7).

The arithmetic of a seminormal finitely primary monoid was studied in [16, 18], and the following lemma describes a relationship between the seminormal closure and the root-closure of finitely primary monoids.

Lemma 3.1. *Let $H \subseteq F = F^\times \times [p_1, \ldots, p_s]$ be a finitely primary monoid of rank s and exponent α, where $p_1, \ldots, p_s$ are pairwise non-associated prime elements of F. Then*

$$\widetilde{H} \setminus (\widetilde{H})^\times = H' \setminus (H')^\times = (p_1 \cdots p_s)F\,,$$

and $\widetilde{H}$ is a root-closed finitely primary monoid of rank s and exponent 1 with its complete integral closure $\widehat{H} = F$. Moreover, $\widetilde{H}$ is a dense C-monoid defined in F.

Proof. Let $x \in \widetilde{H} \setminus (\widetilde{H})^\times$. Then $x^n \in H$ for some $n \in \mathbb{N}$. Since $H^\times = (\widetilde{H})^\times \cap H$ [10, Proposition 1], it follows that

$$x^n \in H \setminus H^\times \subseteq (p_1 \cdots p_s)F\,,$$

and so we infer that

$$x^{n\alpha},\, x^{n\alpha+1} \in (p_1 \cdots p_s)^\alpha F \subseteq H\,,$$

where inclusions follow from the definition of H being a finitely primary monoid. Hence, there exists $N \in \mathbb{N}$ such that any integer $\ell \geq N$ can be written as a non-negative linear combination of integers $n\alpha$ and $n\alpha+1$. Thus, it follows that $x^\ell \in H$ for all $\ell \geq N$, whence $x \in H' \setminus (H')^\times$. Since $(p_1 \cdots p_s)F \subseteq \widetilde{H} \setminus (\widetilde{H})^\times$, the assertion follows by [16, Lemma 3.4.1]. Therefore, we have that

$$\widetilde{H} = (p_1 \cdots p_s)F \cup (\widetilde{H})^\times\,,$$

and it means that $\widetilde{H}$ is a root-closed finitely primary monoid of rank s and exponent 1 such that the complete integral closure of $\widetilde{H}$ is $\widehat{H} = F$, because $H \subseteq \widetilde{H} \subseteq \widehat{H} = F$. Moreover, if we take $V = F^\times$ and $\alpha = 1$, then $\widetilde{H}$ satisfies two conditions described in [15, Corollary 2.9.8], and thus $\widetilde{H}$ is a C-monoid defined in F. Since $\mathsf{v}_{p_i}(H) \subseteq \mathbb{N}_0$ is a numerical monoid for all p_i, it follows that $\widetilde{H}$ is dense in F. ☐

If $\mathcal{C}$ is a semigroup, then a maximal subgroup of $\mathcal{C}$ is constructed by an idempotent element via Green's relation on $\mathcal{C}$ [21, Corollary 4.5]. Thus, idempotent elements of the semigroup $\mathcal{C}$ play a central role in the study of the subgroup structure of $\mathcal{C}$, and so we start with the following observation of idempotent elements in the class semigroup of general C-monoids.

Lemma 3.2. *Let H be a dense C-monoid defined in a factorial monoid $F = F^\times \times \mathcal{F}(P)$, $\mathcal{C} = \mathcal{C}^*(H, F)$ be the reduced class semigroup of H in F, and $a \in F$.*

1. *If $[a]_H^F \in \mathsf{E}(\mathcal{C})$, then $a \in \widehat{H}$, in particular, $a \in H$ if H is finitely generated.*
2. *Suppose that H is seminormal. Then $\big\{[x]_H^F \mid x \in H\big\} \subseteq \mathsf{E}(\mathcal{C})$, and the equality holds if H is finitely generated.*
3. *If H is completely integrally closed, then $[a]_H^F \in \mathsf{E}(\mathcal{C})$ if and only if $a \in H$ if and only if $[a]_H^F = [1]_H^F$.*

Proof. 1. Let $a \in F$ be such that $[a]_H^F \in \mathsf{E}(\mathcal{C})$. Then $a = \varepsilon p_1^{k_1} \cdots p_t^{k_t}$, where $\varepsilon \in F^\times$ and $p_1, \ldots, p_t \in P$. Let $\alpha \in \mathbb{N}$ be an integer and $V \subseteq F^\times$ be a subgroup, satisfying (2.1) and (2.2). Let $i \in [1, t]$. Since H is dense in F, there exists $u \in H$ such that $p_i \mid u$ in F. Then, in view of (2.2), there exists $a \in H$ such that $p_i^\alpha \mid a$ in F, and hence $p_i^\alpha a \in H$. Thus $p_i^\alpha = a^{-1}(p_i^\alpha a) \in \mathsf{q}(H) \cap F = \widehat{H}$. It follows that, for each $i \in [1, t]$, there exists $c_i \in H$ such that $c_i p_i^{\alpha n} \in H$ for all $n \geq 1$. Put $c = c_1 \cdots c_t \in H$. In view of (2.1),

$$ca^\alpha = (c_1 \cdots c_t)\varepsilon^\alpha p_1^{\alpha k_1} \cdots p_t^{\alpha k_t} = \epsilon^\alpha (c_1 p_1^{\alpha k_1}) \cdots (c_t p_t^{\alpha k_t}) \in V(H \setminus H^\times) \subseteq H\,,$$

Since $[a]_H^F \in \mathsf{E}(\mathcal{C})$, $[a]_H^F = [a^n]_H^F$ for all $n \geq 1$, so that $[ca^n]_H^F = [c]_H^F + [a^n]_H^F = [c]_H^F + [a]_H^F = [c]_H^F + [a^\alpha]_H^F = [ca^\alpha]_H^F$ for all $n \geq 1$. Since $1(ca^\alpha) = ca^\alpha \in H$, we infer that $ca^n = 1(ca^n) \in H$ for all $n \geq 1$, whence $a \in \widehat{H}$. In particular, if H is finitely generated, then $\widehat{H} = \widetilde{H}$ (see [15, Proposition 2.7.11]), whence $a^N \in H$

for some $N \in \mathbb{N}$. Since $[a]_H^F \in \mathsf{E}(\mathcal{C})$, $[a]_H^F = [a^N]_H^F$, and thus we infer by the same argument as used before that $a \in H$.

2. Suppose that $H = H'$. If follows by [19, Theorem 1.1] that $\{[x]_H^F \mid x \in H\} \subseteq \mathsf{E}(\mathcal{C})$. Assume, in addition, that H is finitely generated. If $[y]_H^F \in \mathsf{E}(\mathcal{C})$ for $y \in F$, then item 1. ensures that $y \in H$, whence $\mathsf{E}(\mathcal{C}) \subseteq \{[x]_H^F \mid x \in H\}$.

3. Suppose that $H = \widehat{H}$. Then $H = H' = \widehat{H}$, and hence the first equivalent condition follows from items 1. and 2. For the second equivalent condition, assume that $a \in H$. If $a \in H^\times$, then it is obvious that $[a]_H^F = [1]_H^F$. If $a \in H \setminus H^\times$, then for $x \in F$, $ax \in H$ ensures that $1x = x \in \mathsf{q}(H) \cap F = \widehat{H} = H$. Therefore, $a \in H$ is equivalent to $[a]_H^F = [1]_H^F$. □

For the next remark, let us give a brief introduction of the concept of product-one sequences over finite groups. Let G be a finite group with identity 1_G, and $\mathcal{F}(G)$ denote the free abelian monoid with basis G. An element $S = g_1 \cdot \ldots \cdot g_\ell$ of $\mathcal{F}(G)$ is said to be a *product-one sequence* over G if $1_G \in \pi(S) = \{g_{\sigma(1)} \cdots g_{\sigma(\ell)} \in G \mid \sigma \text{ is a permutation of } [1,\ell]\}$, i.e., its terms can be ordered such that their product equals 1_G. The monoid $\mathcal{B}(G)$ of all product-one sequences over G is a finitely generated C-monoid (see [3, Theorem 3.2]), and specific examples of the reduced class semigroup of $\mathcal{B}(G)$ for some non-abelian groups G are provided in [26, Section 4]. We refer the reader to [6] for a recent progress of the algebraic and arithmetic studies over arbitrary groups.

Remark 3.3. Although H is a finitely generated C-monoid, an element $[a]$ with $a \in H$ in the reduced class semigroup of H need not be an idempotent element. To give an example, let G be a finite group with commutator subgroup $G^{(1)}$. Then, $\widehat{\mathcal{B}(G)} = \{S \in \mathcal{F}(G) \mid \pi(S) \subseteq G^{(1)}\}$ (see [13, Proposition 3.1]), and for $S \in \mathcal{F}(G)$, $[S]_{\mathcal{B}(G)}^{\mathcal{F}(G)}$ is an idempotent element in the reduced class semigroup of $\mathcal{B}(G)$ if and only if $\pi(S) \subseteq G^{(1)}$ is a subgroup (see [25, Proposition 3.3]). If $G = \langle \alpha, \beta \mid \alpha^5 = \beta^2 = 1_G \text{ and } \beta\alpha = \alpha^{-1}\beta \rangle$ is a dihedral group of order 10, then $S = \beta \cdot \alpha^2\beta \cdot \alpha^2$ is a product-one sequence over G, but $\pi(S) = \{1_G, \alpha, \alpha^4\} \subset \langle \alpha \rangle$ is not a subgroup. Thus, $[S]_{\mathcal{B}(G)}^{\mathcal{F}(G)}$ is not an idempotent element in the reduced class semigroup of $\mathcal{B}(G)$. Moreover, $\pi(\beta \cdot \alpha^2\beta \cdot \alpha) = \langle \alpha \rangle \setminus \{1_G\}$ ensures that $T = \beta \cdot \alpha^2\beta \cdot \alpha \in \mathcal{B}(G)'$, but $[T]_{\mathcal{B}(G)}^{\mathcal{F}(G)}$ is not an idempotent element in the reduced class semigroup of $\mathcal{B}(G)$.

Theorem 3.4. *Let $H \subseteq F = F^\times \times [p_1, \ldots, p_s]$ be a root-closed finitely primary monoid of rank s, where $p_1, \ldots, p_s$ are pairwise non-associated prime elements of F. Then, every element in the reduced class semigroup is an idempotent element, i.e., $\mathcal{C}^*(H,F) = \mathcal{C} = \mathsf{E}(\mathcal{C})$. More precisely,*

$$\mathcal{C} = \left\{ \left[p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}\right]_H^F \mid r_i \in \{0,1\} \text{ for all } i \in [1,s] \right\} \text{ and } |\mathcal{C}| = 2^s .$$

Proof. By Lemma 3.1, we have that $H = (p_1 \cdots p_s)F \cup H^\times$ and $\widehat{H} = F$. Let $p \in F$ be a prime element. We assert that, for every $x \in F$, $xp \in H$ if and only if $xp^2 \in H$. Let $x \in F$. If $xp \in H = (p_1 \cdots p_s)F \cup H^\times$, then it is obvious that $xp^2 \in H$. Conversely, if $xp^2 \in H$, then for each p_j non-associated with p, we have that $\mathsf{v}_{p_j}(x) \ge 1$, so that $\mathsf{v}_{p_j}(xp) \ge 1$. Thus, we infer that $\mathsf{v}_{p_i}(xp) \ge 1$ for every p_i, and hence $xp \in H$. Therefore, $[p]_H^F = [p^2]_H^F$ for every prime element $p \in F$. Now, if $y = \varepsilon z$ is a non-unit element of F, where $\varepsilon \in F^\times$ and $z \in F \setminus F^\times$, then since $H \setminus H^\times = (p_1 \cdots p_s)F$, we infer that $[y]_H^F = [z]_H^F$. Since every non-unit of F can be written as a product of prime elements of F and $[p]_H^F \in \mathsf{E}(\mathcal{C})$ for every prime $p \in F$, it follows that every element in $\mathcal{C}$ is an idempotent element of the form $[p_1^{r_1} \cdots p_s^{r_s}]_H^F$ for $r_1, \ldots, r_s \in \{0,1\}$. □

Every class in the reduced class semigroup need not be an idempotent element for a general finitely primary monoid as the next simple example shows.

Example 3.5. Let $H = p_1^2 p_2 F \cup \{1\} \subseteq F = \mathcal{F}(\{p_1, p_2\})$ be a finitely primary monoid of rank 2 and exponent 2. If we take $V = \{1\}$ and $\alpha = 2$, then H satisfies two conditions described in [15, Corollary

2.9.8], whence H is a C-monoid. Since $p_1p_2 \notin H$, it follows that $H \subsetneq H' = \widetilde{H} = p_1p_2F$ by Lemma 3.1. Moreover, $(p_1p_2)^2 = (p_1^2p_2)p_2 \in H$ implies that $[p_1p_2]_H^F \neq [(p_1p_2)^2]_H^F$, whence $[p_1p_2]_H^F$ is not an idempotent element in the reduced class semigroup of H in F.

Let H be a root-closed finitely primary monoid. Since every root-closed monoid is a seminormal monoid, it follows by [19, Theorem 1.1] that the reduced class semigroup of H is a Clifford semigroup, i.e., it is a union of its subgroups. Moreover, Theorem 3.4 ensures that every singleton set is a maximal subgroup of the reduced class semigroup of H, which is actually the partial Ponizovsky factor (see [21, Chapter IV]).

Corollary 3.6. *Let $H \subseteq F = F^\times \times [p_1, \ldots, p_s]$ be a root-closed finitely primary monoid of rank s, where $p_1, \ldots, p_s$ are pairwise non-associated prime elements of F, and $\mathcal{C} = \mathcal{C}^*(H,F)$. Then, for each $i \in [1,s]$, $\mathcal{C}_i = \big\{[p_i]_H^F, [1]_H^F\big\}$ is a subsemigroup of $\mathcal{C}$, and there exists a semigroup isomorphism $\mathcal{C} \cong \prod_{i\in[1,s]} \mathcal{C}_i$.*

Proof. For each $i \in [1,s]$, $[p_i]_H^F \in \mathsf{E}(\mathcal{C})$ by Theorem 3.4, and hence it is obvious that $\mathcal{C}_i = \big\{[p_i]_H^F, [1]_H^F\big\}$ is a subsemigroup of $\mathcal{C}$. Now, define the map

$$\theta : \mathcal{C} \to \mathcal{C}_1 \times \cdots \times \mathcal{C}_s \quad \text{by} \quad \theta\big([x]_H^F\big) \;=\; \big([p_1^{r_1}]_H^F, \ldots, [p_s^{r_s}]_H^F\big)\,,$$

where $x = \varepsilon p_1^{r_1} \cdots p_s^{r_s} \in F$ with $\varepsilon \in F^\times$ and $r_1, \ldots, r_s \in \mathbb{N}_0$. Then, we may assume by Theorem 3.4 that $r_1, \ldots, r_s \in \{0,1\}$, and hence $\theta\big([x]_H^F\big) \in \prod_{i\in[1,s]} \mathcal{C}_i$. As a direct consequence of Theorem 3.4, we infer that θ is a well-defined bijection. If $x = \varepsilon p_1^{r_1} \cdots p_s^{r_s}$ and $y = \delta p_1^{k_1} \cdots p_s^{k_s}$ for $r_1, \ldots, r_s, k_1, \ldots, k_s \in \{0,1\}$ not all zero, then $[xy]_H^F = [p_1^{\ell_1} \cdots p_s^{\ell_s}]_H^F$, where $r_i + k_i \equiv \ell_i \pmod 2$ for all $i \in [1,s]$, so that

$$\theta\big([x]_H^F + [y]_H^F\big) \;=\; \theta\big([xy]_H^F\big) \;=\; \big([p_1^{r_1}]_H^F, \ldots, [p_s^{r_s}]_H^F\big) + \big([p_1^{k_1}]_H^F, \ldots, [p_s^{k_s}]_H^F\big) \;=\; \theta\big([x]_H^F\big) + \theta\big([y]_H^F\big)\,,$$

whence θ is a semigroup isomorphism. □

We end this section with the algebraic structure of the reduced class semigroup of a large class of finitely primary monoids that are not root-closed.

Theorem 3.7. *Let $k_1, \ldots, k_s \in \mathbb{N}$, $H = p_1^{k_1} \cdots p_s^{k_s} F \cup H^\times \subseteq F = F^\times \times [p_1, \ldots, p_s]$ be a finitely primary monoid of rank s, where $p_1, \ldots, p_s$ are pairwise non-associated prime elements of F, and $\mathcal{C} = \mathcal{C}^*(H,F)$.*

1. *For each $i \in [1,s]$, $[p_i^{k_i}]_H^F = [p_i^{k_i+1}]_H^F$, and in particular, $[p_i^{k_i}]_H^F$ is an idempotent element in $\mathcal{C}$.*
2. *$\mathcal{C} = \big\{[p_1^{r_1} \cdots p_s^{r_s}]_H^F \mid r_i \in [0, k_i] \text{ for all } i \in [1,s]\big\}$ and $|\mathcal{C}| = \prod_{i\in[1,s]}(k_i + 1)$.*
3. *For each $i \in [1,s]$, $\mathcal{C}_i = \big\{[p_i]_H^F, \ldots, [p_i^{k_i}]_H^F, [1]_H^F\big\}$ is a subsemigroup of $\mathcal{C}$, and there exists a semigroup isomorphism $\mathcal{C} \cong \prod_{i\in[1,s]} \mathcal{C}_i$.*

Proof. 1. Let $i \in [1,s]$, and $x \in F$. If $xp_i^{k_i} \in H$, then it is obvious that $xp_i^{k_i+1} \in H$. If $xp_i^{k_i+1} \in H$, then $\mathsf{v}_{p_i}(x) \geq 0$ and $\mathsf{v}_{p_j}(x) \geq k_j$ for every $j \neq i$, whence $xp_i^{k_i} \in H$. Thus, $[p_i^{k_i}]_H^F = [p_i^{k_i+1}]_H^F$, and thus,

$$[p_i^{k_i+2}]_H^F \;=\; [p_i^{k_i+1}]_H^F + [p_i]_H^F \;=\; [p_i^{k_i}]_H^F + [p_i]_H^F \;=\; [p_i^{k_i+1}]_H^F \;=\; [p_i^{k_i}]_H^F\,.$$

By the inductive argument, we infer that $[p_i^{2k_i}]_H^F = [p_i^{k_i}]_H^F$, whence $[p_i^{k_i}]_H^F \in \mathsf{E}(\mathcal{C})$.

2. Let $x = \varepsilon p_1^{r_1} \cdots p_s^{r_s}, y = \delta p_1^{\ell_1} \cdots p_s^{\ell_s} \in F$ for some $\varepsilon, \delta \in F^\times$ and $r_1, \ldots, r_s, \ell_1, \ldots, \ell_s \in \mathbb{N}_0$ not all zero. We assert that $[x]_H^F = [y]_H^F$ if and only if $r_i \equiv \ell_i \pmod{k_i}$ for all $i \in [1,s]$. If $r_i \equiv \ell_i \pmod{k_i}$ for all $i \in [1,s]$, then it is clear that $[x]_H^F = [y]_H^F$. Suppose now that $[x]_H^F = [y]_H^F$. Then, item 1. ensures that each r_i and ℓ_i can be reduced by modulo k_i, and thus we can assume that $r_i, \ell_i \in [0, k_i]$, not all zero, for every $i \in [1,s]$. If $r_i \neq \ell_i$ for some $i \in [1,s]$, then we may assume that $r_i \lneq \ell_i$, and so we can choose $n \geq 0$ such that $r_i + n \lneq k_i \leq \ell_i + n$. If $z \in F$ is an element such that $\mathsf{v}_{p_i}(z) = n$ and $\mathsf{v}_{p_j}(z) = k_j$ for every $j \neq i$, then $zy \in H$, but $zx \notin H$, a contradiction. Thus, $r_i = \ell_i$ for all $i \in [1,s]$, and therefore the assertion follows.

3. Let $i \in [1,s]$. Then, item 1. implies that $\mathcal{C}_i = \big\{[p_i]_H^F, \ldots, [p_i^{k_i}]_H^F, [1]_H^F\big\}$ is a subsemigroup of $\mathcal{C}$. Now we define the map

$$\theta \colon \mathcal{C} \to \mathcal{C}_1 \times \cdots \times \mathcal{C}_s \quad \text{by} \quad \theta\big([x]_H^F\big) \;=\; \big([p_1^{r_1}]_H^F, \ldots, [p_s^{r_s}]_H^F\big)\,,$$

where $x = \varepsilon p_1^{r_1} \cdots p_s^{r_s} \in F$ with $\varepsilon \in F^\times$ and $r_1, \ldots, r_s \in \mathbb{N}_0$ not all zero. Then, by item 2., we may assume that $r_i \in [0, k_i]$, not all zero, for every $i \in [1, s]$, so that $\theta\big([x]_H^F\big) \in \prod_{i\in[1,s]} \mathcal{C}_i$ and θ is a well-defined bijection. If $x = \varepsilon p_1^{r_1} \cdots p_s^{r_s}$ and $y = \delta p_s^{\ell_1} \cdots p_s^{\ell_s}$ with $\varepsilon, \delta \in F^\times$ and $r_i, \ell_i, \in [0, k_i]$, not all zero, for all $i \in [1, s]$, then in view of r_i, ℓ_i as elements of a cyclic group $\mathbb{Z}_{k_i}$ modulo k_i, it follows that

$$\theta\big([x]_H^F + [y]_H^F\big) = \theta\big([xy]_H^F\big) = \big([p_1^{r_1}]_H^F, \ldots, [p_s^{r_s}]_H^F\big) + \big([p_1^{\ell_1}]_H^F, \ldots, [p_s^{\ell_s}]_H^F\big) = \theta\big([x]_H^F\big) + \theta\big([y]_H^F\big),$$

whence θ is a semigroup isomorphism. □

4. The root-closed weakly Krull Mori monoids

In this section, we study the algebraic structure of the reduced class semigroup of root-closed weakly Krull Mori monoids. Our main references are [15, 22]. Let H be a monoid. An element $q \in H$ is said to be *primary* if $q \notin H^\times$, and for all $a, b \in H$, $q \mid ab$ implies that $q \mid a$ or $q \mid b^n$ for some $n \in \mathbb{N}$. The monoid H is called *primary* if $H \neq H^\times$ and every non-unit is primary. Every finitely primary monoid is primary, and every saturated submonoid of a primary monoid is again primary. The monoid H is said to be *weakly factorial* if every non-unit element can be written as a product of primary elements. Every primary monoid is weakly factorial, and every coproduct of a weakly factorial monoid is again weakly factorial.

Let $\mathfrak{X}(H)$ be the set of non-empty minimal prime ideals of H. For $\mathfrak{p} \in \mathfrak{X}(H)$, we denote by $H_\mathfrak{p} = (H \setminus \mathfrak{p})^{-1}H \subseteq \mathsf{q}(H)$ the localization of H at $\mathfrak{p}$. The monoid H is said to be *weakly Krull* [22, Corollary 22.5] if

$$H = \bigcap_{\mathfrak{p}\in\mathfrak{X}(H)} H_\mathfrak{p} \quad \text{and} \quad \{\mathfrak{p} \in \mathfrak{X}(H) \mid a \in \mathfrak{p}\} \ \text{is finite for all} \ a \in H\,.$$

If H is a weakly Krull monoid, then the family of embeddings $(\varphi_\mathfrak{p} \colon H \hookrightarrow H_\mathfrak{p})_{\mathfrak{p}\in\mathfrak{X}(H)}$ induces a divisor homomorphism $\varphi \colon H \to \coprod_{\mathfrak{p}\in\mathfrak{X}(H)} (H_\mathfrak{p})_{\mathrm{red}}$ given by $\varphi(a) = (aH_\mathfrak{p}^\times)_{\mathfrak{p}\in\mathfrak{X}(H)}$ [15, Proposition 2.6.2]. Note that $H_\mathfrak{p}$ is a primary monoid for every $\mathfrak{p} \in \mathfrak{X}(H)$, and a weakly Krull monoid is Krull if and only if $H_\mathfrak{p}$ is a discrete valuation monoid, i.e., $(H_\mathfrak{p})_{\mathrm{red}} \cong \mathbb{N}_0$, for all $\mathfrak{p} \in \mathfrak{X}(H)$. If H is Mori, then H is weakly factorial if and only if H is weakly Krull and $\mathcal{C}_v(H) = \{0\}$ (see, [22, Exercise 5 on p. 258]).

A domain R is said to be a *weakly Krull domain* if $R^\bullet$ is a weakly Krull monoid. Weakly Krull domains generalize one-dimensional Noetherian domains, but they need not be integrally closed. For instance, every order in a number field is a weakly Krull domain (in particular, the principal order is a Krull domain). Let R be a domain, and H be a torsionless monoid such that $\mathsf{q}(H)$ is torsion-free. Then, the monoid algebra $R[H]$ is root-closed if and only if both R and H are root-closed [1, Corollary 2.5], and as a recent result, we refer the reader to [4] for a characterization of when a monoid algebra is weakly Krull. Clearly, every Krull monoid is a root-closed weakly Krull Mori monoid, and the algebraic and arithmetic properties are well-studied for a Krull monoid.

We start with the following basic properties of root-closed monoids, and the seminormal analogues can be found in [16, Lemma 3.2].

Lemma 4.1. *Let F be a monoid.*

1. *If $S \subseteq F$ is a submonoid, then $\widetilde{S^{-1}F} = S^{-1}\widetilde{F}$ and $(S^{-1}F)' = S^{-1}F'$. Furthermore, if F is root-closed (resp., seminormal), then $S^{-1}F$ is root-closed (resp., seminormal).*
2. *If $(F_i)_{i\in I}$ is a family of monoids such that $F = \coprod_{i\in I} F_i$, then $\widetilde{F} = \coprod_{i\in I} \widetilde{F_i}$ and $F' = \coprod_{i\in I} F_i'$. In particular, F is root-closed (resp., seminormal) if and only if F_i is root-closed (resp., seminormal) for all $i \in I$.*
3. *$\widetilde{F_{\mathrm{red}}} = \widetilde{F}/F^\times$ and $(F_{\mathrm{red}})' = F'/F^\times$, and in particular, F is root-closed (resp., seminormal) if and only if F_{red} is root-closed (resp., seminormal).*

4. *If F is root-closed (resp., seminormal) and $H \subseteq F$ is a saturated submonoid, then H is root-closed (resp., seminormal).*

Proof. We prove the statements only for the root-closed case.

1. Let $S \subseteq F$ be a submonoid, and $x \in \mathsf{q}(S^{-1}F) = \mathsf{q}(F)$ be such that $x^n \in S^{-1}F$ for some $n \in \mathbb{N}$. Then, there exists $s \in S$ such that $sx^n \in F$, so that $(sx)^n \in F$. It follows that $sx \in \widetilde{F}$, and thus $x \in S^{-1}\widetilde{F}$. For the reverse containment, if $x \in S^{-1}\widetilde{F}$, then there exist $s \in S$ and $n \in \mathbb{N}$ such that $(sx)^n \in F$. Thus, we have that $x^n \in S^{-1}F$, so that $x \in \widetilde{S^{-1}F}$, whence the assertion follows. Furthermore, if F is root-closed, then $\widetilde{S^{-1}F} = S^{-1}\widetilde{F} = S^{-1}F$, and thus $S^{-1}F$ is root-closed.

2. It is easy to be verified from $\mathsf{q}(F) = \coprod_{i\in I} \mathsf{q}(F_i)$.

3. Let $\varphi : \mathsf{q}(F) \to \mathsf{q}(F)/F^\times = \mathsf{q}(F_{\mathrm{red}})$ be the canonical epimorphism. Then $\varphi|_F : F \to F_{\mathrm{red}}$ is surjective, and hence, if $x \in \mathsf{q}(F)$ and $n \in \mathbb{N}$, then $x^n \in F$ if and only if $\varphi(x)^n \in F_{\mathrm{red}}$. Thus, it follows that $x \in F'$ (resp., $x \in \widetilde{F}$) if and only if $\varphi(x) \in (F_{\mathrm{red}})'$ (resp., $\varphi(x) \in \widetilde{F_{\mathrm{red}}}$). As submonoids of $\mathsf{q}(F_{\mathrm{red}})$, we infer that $(F_{\mathrm{red}})' = F'/F^\times$ and $\widetilde{F_{\mathrm{red}}} = \widetilde{F}/F^\times$.

4. Let F be a root-closed monoid, and $H \subseteq F$ be a saturated submonoid. If $x \in \mathsf{q}(H) \subseteq \mathsf{q}(F)$ is such that $x^n \in H \subseteq F$ for some $n \in \mathbb{N}$, then since F is root-closed and $H \subseteq F$ is saturated, $x \in \mathsf{q}(H) \cap F = H$. Thus, H is root-closed. ☐

Next, we show that the localization of a weakly Krull monoid at a minimal prime preserves the root-closedness, and the seminormal and Mori analogues can be found in [16, Proposition 5.3].

Lemma 4.2. *Let H be a weakly Krull monoid. Then H is root-closed (resp., seminormal, or Mori) if and only if $H_\mathfrak{p}$ is root-closed (resp., seminormal, or Mori) for each $\mathfrak{p} \in \mathfrak{X}(H)$.*

Proof. We prove the statements only for the root-closed case. ($\Rightarrow$) This follows by Lemma 4.1.1. ($\Leftarrow$) Suppose that $H_\mathfrak{p}$ is root-closed for each $\mathfrak{p} \in \mathfrak{X}(H)$. Then, by Lemma 4.1.2, the coproduct $\coprod_{\mathfrak{p}\in\mathfrak{X}(H)} H_\mathfrak{p}$ is root-closed. Since H is weakly Krull, there is a divisor homomorphism from H to $\coprod_{\mathfrak{p}\in\mathfrak{X}(H)} H_\mathfrak{p}$, and it follows that H_{red} is isomorphic to a saturated submonoid of $\coprod_{\mathfrak{p}\in\mathfrak{X}(H)} H_\mathfrak{p}$. By Lemma 4.1.4, H_{red} is also root-closed, and therefore, H is root-closed by Lemma 4.1.3. ☐

Proposition 4.3. *Let H be a weakly Krull Mori monoid with $\emptyset \neq \mathfrak{f} = (H : \widehat{H}) \subsetneq H$ such that $H_\mathfrak{p}$ is finitely primary for each $\mathfrak{p} \in \mathfrak{X}(H)$.*

1. *$\widehat{H}$ is Krull, $P^* = \{\mathfrak{p} \in \mathfrak{X}(H) \mid \mathfrak{f} \subseteq \mathfrak{p}\}$ is finite, for each $\mathfrak{p} \in \mathfrak{X}(H) \setminus P^*$, $H_\mathfrak{p}$ is a discrete valuation monoid.*
2. *$\mathcal{I}_v^*(H) \cong \mathcal{F}(P) \times \prod_{\mathfrak{p}\in P^*} (H_\mathfrak{p})_{\mathrm{red}}$, where $P = \mathfrak{X}(H) \setminus P^*$, is a weakly factorial Mori monoid.*

Proof. 1. Since H is a Mori monoid, the assertion follows by [15, Theorems 2.2.5 and 2.6.5].

2. [16, Theorem 5.3.4]. ☐

Now, we give the main result of this paper concerning the algebraic structure of the reduced class semigroup of a root-closed weakly Krull Mori monoid.

Theorem 4.4. *Let H be a root-closed weakly Krull Mori monoid such that $\emptyset \neq \mathfrak{f} = (H : \widehat{H}) \subsetneq H$ and $H_\mathfrak{p}$ is finitely primary for each $\mathfrak{p} \in \mathfrak{X}(H)$. Assume that $\widehat{H}_\mathfrak{p}^\times / H_\mathfrak{p}^\times$ is finite for each $\mathfrak{p} \in P^* = \{\mathfrak{p} \in \mathfrak{X}(H) \mid \mathfrak{f} \subseteq \mathfrak{p}\}$.*

1. *$\mathcal{I}_v^*(H)$ is a C-monoid defined in $\widehat{\mathcal{I}_v^*(H)}$, and there exists a semigroup isomorphism*

$$\mathcal{C}^*\big(\mathcal{I}_v^*(H), \widehat{\mathcal{I}_v^*(H)}\big) \cong \prod_{\mathfrak{p}\in P^*} \mathcal{C}^*(H_\mathfrak{p}, \widehat{H}_\mathfrak{p}) \cong \prod_{\mathfrak{p}\in P^*} \big(\mathcal{C}_1 \times \cdots \times \mathcal{C}_{s_\mathfrak{p}}\big)\,,$$

where for each $\mathfrak{p} \in P^*$, $s_{\mathfrak{p}} = \left|\{\mathfrak{P} \in \mathfrak{X}(\widehat{H}) \mid \mathfrak{P} \cap H = \mathfrak{p}\}\right|$, $\mathcal{C}_i = \left\{[\mathfrak{P}_i(\mathfrak{p})]^{\widehat{H}_{\mathfrak{p}}}_{H_{\mathfrak{p}}}, [1]^{\widehat{H}_{\mathfrak{p}}}_{H_{\mathfrak{p}}}\right\}$ *for* $i \in [1, s_{\mathfrak{p}}]$, *and* $\{\mathfrak{P}_1(\mathfrak{p}), \ldots, \mathfrak{P}_{s_{\mathfrak{p}}}(\mathfrak{p})\}$ *is the set of pairwise non-associated prime elements in* $\widehat{H}_{\mathfrak{p}}$.

2. *Suppose that* $\mathcal{C}_v(H)$ *is finite.*
   (a) H_{red} *is a C-monoid defined in* $F = \mathcal{F}(P) \times \prod_{\mathfrak{p} \in P^*} \widehat{H}_{\mathfrak{p}}/H_{\mathfrak{p}}^{\times}$.
   (b) *If* H_{red} *is dense in* F*, then* H *is weakly factorial if and only if* $\widehat{H}$ *is factorial. In this case,* $\mathcal{C}^*(H_{\mathrm{red}}, F) \cong \mathcal{C}^*\big(\mathcal{I}_v^*(H), \widehat{\mathcal{I}_v^*(H)}\big)$.

Proof. 1. By Proposition 4.3.2, there exists an isomorphism

$$\mathcal{I}_v^*(H) \cong \mathcal{F}(P) \times \prod_{\mathfrak{p} \in P^*} (H_{\mathfrak{p}})_{\mathrm{red}}\,, \quad \text{where } P = \mathfrak{X}(H) \setminus P^*\,.$$

Let $\mathfrak{p} \in P^*$. Then, $H_{\mathfrak{p}}$ is root-closed (by Lemma 4.2) and finitely primary of rank $\big|\mathfrak{X}(\widehat{H_{\mathfrak{p}}})\big|$. By [16, Lemma 5.1], $\{\mathfrak{P} \in \mathfrak{X}(\widehat{H}) \mid \mathfrak{P} \cap H = \mathfrak{p}\} = \{\mathfrak{q} \cap \widehat{H} \mid \mathfrak{q} \in \mathfrak{X}(\widehat{H_{\mathfrak{p}}})\}$ is the set of all non-empty minimal prime ideals of $\widehat{H}$ lying above $\mathfrak{p}$, whence $\big|\mathfrak{X}(\widehat{H_{\mathfrak{p}}})\big| = \big|\{\mathfrak{P} \in \mathfrak{X}(\widehat{H}) \mid \mathfrak{P} \cap H = \mathfrak{p}\}\big| = s_{\mathfrak{p}}$. Thus, $H_{\mathfrak{p}}$ is a root-closed finitely primary monoid of rank $s_{\mathfrak{p}}$, and by Lemma 3.1, it is a C-monoid defined in a factorial monoid $\widehat{H_{\mathfrak{p}}}$. Note that $\widehat{H_{\mathfrak{p}}} = \widehat{H}_{\mathfrak{p}}$ (see [15, Theorem 2.3.5]). Since $\widehat{H}_{\mathfrak{p}}^{\times}/H_{\mathfrak{p}}^{\times}$ is finite for each $\mathfrak{p} \in P^*$, [15, Theorem 2.9.16] ensures that $\mathcal{F}(P) \times \prod_{\mathfrak{p} \in P^*} H_{\mathfrak{p}}$ is a C-monoid defined in a factorial monoid $\mathcal{F}(P) \times \prod_{\mathfrak{p} \in P^*} \widehat{H}_{\mathfrak{p}}$, so that $\big(\mathcal{F}(P) \times \prod_{\mathfrak{p} \in P^*} H_{\mathfrak{p}}\big)_{\mathrm{red}}$ is also a C-monoid defined in $\mathcal{F}(P) \times \prod_{\mathfrak{p} \in P^*} \widehat{H}_{\mathfrak{p}}/H_{\mathfrak{p}}^{\times}$ by [15, Theorem 2.9.10]. Then, since $\widehat{(H_{\mathfrak{p}})_{\mathrm{red}}} = \widehat{H_{\mathfrak{p}}}/H_{\mathfrak{p}}^{\times} = \widehat{H}_{\mathfrak{p}}/H_{\mathfrak{p}}^{\times}$, it follows that $\widehat{\mathcal{I}_v^*(H)} \cong \mathcal{F}(P) \times \prod_{\mathfrak{p} \in P^*} \widehat{(H_{\mathfrak{p}})_{\mathrm{red}}} = \mathcal{F}(P) \times \prod_{\mathfrak{p} \in P^*} \widehat{H}_{\mathfrak{p}}/H_{\mathfrak{p}}^{\times}$, whence $\mathcal{I}_v^*(H)$ is a C-monoid defined in $\widehat{\mathcal{I}_v^*(H)}$.

By [15, Lemmas 2.8.6 and 2.8.4], we infer that

$$\mathcal{C}^*\big(\mathcal{I}_v^*(H), \widehat{\mathcal{I}_v^*(H)}\big) \cong \prod_{\mathfrak{p} \in P^*} \mathcal{C}^*\big((H_{\mathfrak{p}})_{\mathrm{red}}, \widehat{(H_{\mathfrak{p}})_{\mathrm{red}}}\big) = \prod_{\mathfrak{p} \in P^*} \mathcal{C}^*\big(H_{\mathfrak{p}}/H_{\mathfrak{p}}^{\times}, \widehat{H}_{\mathfrak{p}}/H_{\mathfrak{p}}^{\times}\big) \cong \prod_{\mathfrak{p} \in P^*} \mathcal{C}^*(H_{\mathfrak{p}}, \widehat{H}_{\mathfrak{p}})\,.$$

For each $\mathfrak{p} \in P^*$, since $H_{\mathfrak{p}} \subseteq \widehat{H}_{\mathfrak{p}}$ is root-closed finitely primary of rank $s_{\mathfrak{p}}$, it follows by Corollary 3.6 that

$$\mathcal{C}^*(H_{\mathfrak{p}}, \widehat{H}_{\mathfrak{p}}) \cong \mathcal{C}_1 \times \cdots \times \mathcal{C}_{s_{\mathfrak{p}}}\,,$$

where $\mathcal{C}_i = \left\{[\mathfrak{P}_i(\mathfrak{p})]^{\widehat{H}_{\mathfrak{p}}}_{H_{\mathfrak{p}}}, [1]^{\widehat{H}_{\mathfrak{p}}}_{H_{\mathfrak{p}}}\right\}$ is a subsemigroup of $\mathcal{C}^*(H_{\mathfrak{p}}, \widehat{H}_{\mathfrak{p}})$ for each $i \in [1, s_{\mathfrak{p}}]$, and $\big(\widehat{H}_{\mathfrak{p}}\big)_{\mathrm{red}} \cong \big[\mathfrak{P}_1(\mathfrak{p}), \ldots, \mathfrak{P}_{s_{\mathfrak{p}}}(\mathfrak{p})\big]$ with pairwise non-associated prime elements $\mathfrak{P}_1(\mathfrak{p}), \ldots, \mathfrak{P}_{s_{\mathfrak{p}}}(\mathfrak{p})$ in $\widehat{H}_{\mathfrak{p}}$.

2.(a) Since $\mathcal{I}_v^*(H)/\mathcal{H}(H) = \mathcal{C}_v(H)$ is finite, $\mathcal{H}(H)$ is a C-monoid defined in $\widehat{\mathcal{I}_v^*(H)}$ by [15, Theorem 2.9.10]. Let $F = \mathcal{F}(P) \times \prod_{\mathfrak{p} \in P^*} \widehat{H}_{\mathfrak{p}}/H_{\mathfrak{p}}^{\times}$. Since $H_{\mathrm{red}} \cong \mathcal{H}(H)$ and $F \cong \widehat{\mathcal{I}_v^*(H)}$, we infer that H_{red} is a C-monoid defined in F,

2.(b) ($\Rightarrow$) Suppose that H is weakly factorial. By [16, Proposition 5.4], we infer that there exists an epimorphism $\varphi : \mathcal{C}_v(H) \to \mathcal{C}_v(\widehat{H})$ given by $\varphi([\mathfrak{a}]) = [\mathfrak{a}_{v(\widehat{H})}]$, where $\mathfrak{a} \in \mathcal{I}_v^*(H)$. Since H is a weakly Krull Mori monoid, it follows that $\mathcal{C}_v(H) = \{0\}$, and thus $\widehat{H}$ is a Krull monoid (by Proposition 4.3) with $\mathcal{C}_v(\widehat{H}) = \{0\}$. Hence, $\widehat{H}$ is factorial.

($\Leftarrow$) Suppose that $\widehat{H}$ is factorial, i.e., H is a Krull monoid with trivial v-class group. Then, $\big(\widehat{H}/H^{\times}\big)_{\mathrm{red}} = \widehat{H}_{\mathrm{red}}$ is a free monoid, so that $\widehat{H_{\mathrm{red}}} = \widehat{H}/H^{\times}$ is also factorial. Thus, $\widehat{H_{\mathrm{red}}}$ is a Krull monoid with $\mathcal{C}_v\big(\widehat{H_{\mathrm{red}}}\big) = \{0\}$. Note that H_{red} is a C-monoid defined in F by 2.(a). Since H_{red} is dense in F, it follows by Lemma 2.1 that $\widehat{H_{\mathrm{red}}}$ is a saturated and cofinal submonoid of F, and there exists a divisor theory from $\widehat{H_{\mathrm{red}}}$ to the non-unit part of a factorial monoid F. By [15, Theorems 2.4.7 and 2.8.7], we have that

$$\mathcal{C}_v\big(\widehat{H_{\mathrm{red}}}\big) \cong F/\widehat{H_{\mathrm{red}}} \cong \mathcal{C}\big(\widehat{H_{\mathrm{red}}}, F\big)\,,$$

and thus $\mathcal{C}\big(\widehat{H_{\mathrm{red}}}, F\big)$ is a trivial semigroup. It means that, for every $x \in F$, $[x]^F_{\widehat{H_{\mathrm{red}}}} = [1]^F_{\widehat{H_{\mathrm{red}}}}$ implies that $x \in \widehat{H_{\mathrm{red}}}$, so that $\widehat{H_{\mathrm{red}}} = F$. If $\mathfrak{a} \in \mathcal{I}_v^*(H)$, then since $H_{\mathrm{red}} \cong \mathcal{H}(H)$ and $F \cong \widehat{\mathcal{I}_v^*(H)}$, we obtain that $\mathfrak{a} \in \widehat{\mathcal{I}_v^*(H)} = \widehat{\mathcal{H}(H)} \subseteq \mathsf{q}\big(\mathcal{H}(H)\big)$. Since $\mathcal{H}(H)$ is saturated in $\mathcal{I}_v^*(H)$, we infer that $\mathfrak{a} \in \mathcal{H}(H)$, whence $\mathcal{H}(H) = \mathcal{I}_v^*(H)$. Therefore, H is a weakly Krull Mori monoid with $\mathcal{C}_v(H) = \{0\}$, so that H is weakly factorial. The remaining assertion follows by item 1. □

The following lemma describes a characterization of when the multiplicative monoid of a domain is root-closed finitely primary. A seminormal analogue can be found in [16, Lemma 3.4].

Lemma 4.5.

1. *A domain R is one-dimensional and local if and only if $R^\bullet$ is a primary monoid.*
2. *The following statements are equivalent for a domain R:*
   (a) *R is a root-closed (resp., seminormal) one-dimensional local Mori domain.*
   (b) *$R^\bullet$ is a root-closed (resp., seminormal) finitely primary monoid.*

Proof. 1. [15, Proposition 2.10.7].

2. (a) ⇒ (b) Suppose that R is a root-closed one-dimensional local Mori domain. By 1., $R^\bullet$ is a primary monoid. We assert that $(R^\bullet : \widehat{R^\bullet}) \neq \emptyset$. Note that $R \setminus R^\times \neq \{0\}$, for otherwise R must be a field, so that R is zero-dimensional, a contradiction. Let $0 \neq a \in R \setminus R^\times$. If $x \in \widehat{R^\bullet}$, then there exists $c \in R^\bullet$ such that $cx^n \in R^\bullet$ for all $n \in \mathbb{N}$. If $c \in R^\times$, then $x^n \in R$ for all $n \in \mathbb{N}$, and in particular, $x \in R^\bullet$. Thus, $ax \in R^\bullet$. If $c \in R \setminus R^\times$, then since $R^\bullet$ is primary, it follows that $c \mid a^k$ for some $k \in \mathbb{N}$, so that $a^k = bc$ for some $b \in R^\bullet$. Thus, $(ax)^k = b(cx^k) \in R$, and since R is root-closed, we infer that $ax \in R^\bullet$. In either case, we obtain that $a \in (R^\bullet : \widehat{R^\bullet})$. Therefore, the assertion follows by [15, Proposition 2.10.7].

(b) ⇒ (a) Since $R^\bullet$ is root-closed finitely primary, it follows by Lemma 3.1 that $R^\bullet$ is a C-monoid, and hence $R^\bullet$ is a Mori monoid [15, Theorem 2.9.13], i.e., R is a root-closed Mori domain. Since every finitely primary monoid is primary, we infer by item 1. that R is a one-dimensional local domain. □

Example 4.6. 1. Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$, K be the subfield of $\overline{\mathbb{Q}}$ consisting of all elements $u \in \overline{\mathbb{Q}}$ such that the minimal polynomial for u over $\mathbb{Q}$ is solvable by radicals over $\mathbb{Q}$, $F = K(\alpha)$, and $V = F[[X]]$, where $\alpha \in \overline{\mathbb{Q}} \setminus K$ and X is an indeterminate over F. Then, $R = K + XV$ is a root-closed one-dimensional local Noetherian (and so, Mori) domain [1, Example 2.2].

2. Let R be a non-principal order in a number field. Then, R is a one-dimensional Noetherian domain with $(R : \widehat{R}) \neq \{0\}$, especially, it is a weakly Krull Mori domain. For each non-zero prime ideal $\mathfrak{p}$ of R, $R_\mathfrak{p}$ is a one-dimensional local Noetherian domain and $\widehat{R}_\mathfrak{p}^\times / R_\mathfrak{p}^\times$ is finite (see [24, Section I.12]). It is known that R is root-closed if and only if $(R : \widehat{R})$ is an intersection of maximal ideals P_i of $\widehat{R}$ such that $\big|\widehat{R}/P_i\big| = 2$ for each P_i (see [27, Corollary 2.2]). Thus, every multiplicative monoid of a root-closed non-principal order in a number field satisfies the hypothesis of Theorem 4.4. In particular, $R = \mathbb{Z}[\sqrt{17}]$ is a root-closed non-principal order in a quadratic number field ([2, Proposition]).

Corollary 4.7. *Let R be a weakly Krull Mori domain with $\{0\} \neq \mathfrak{f} = (R : \widehat{R}) \subsetneq R$, $\mathfrak{X}(R)$ be the set of non-zero minimal prime ideals of R, $P^* = \{\mathfrak{p} \in \mathfrak{X}(R) \mid \mathfrak{f} \subseteq \mathfrak{p}\}$, and $P = \mathfrak{X}(R) \setminus P^*$. For each $\mathfrak{p} \in P^*$, let $s_\mathfrak{p}$ be the number of prime ideals $\widehat{\mathfrak{p}} \in \mathfrak{X}(\widehat{R})$ such that $\widehat{\mathfrak{p}} \cap R = \mathfrak{p}$.*

1. *P^* is finite, and for each $\mathfrak{p} \in P^*$, the monoid $R_\mathfrak{p}^\bullet$ is finitely primary of rank $s_\mathfrak{p}$.*
2. *There exists a monoid isomorphism $\mathcal{I}_v^*(R) \cong \mathcal{F}(P) \times \prod_{\mathfrak{p} \in P^*} (R_\mathfrak{p}^\bullet)_{\mathrm{red}}$ given by $\mathfrak{a} \mapsto (a_\mathfrak{p} R_\mathfrak{p}^\times)_{\mathfrak{p} \in \mathfrak{X}(R)}$ if $\mathfrak{a}_\mathfrak{p} = a_\mathfrak{p} R_\mathfrak{p}$ for all $\mathfrak{p} \in \mathfrak{X}(R)$.*
3. *Suppose that R is root-closed, $\mathcal{C}_v(R)$ is finite, and $(\widehat{R_\mathfrak{p}^\bullet})^\times / (R_\mathfrak{p}^\bullet)^\times$ is finite for all $\mathfrak{p} \in \mathfrak{X}(R)$.*
   (a) *R is a C-domain, in particular, $(R^\bullet)_{\mathrm{red}}$ is a C-monoid defined in $F = \mathcal{F}(P) \times \prod_{\mathfrak{p} \in P^*} \widehat{R_\mathfrak{p}^\bullet} / (R_\mathfrak{p}^\bullet)^\times$.*

(b) *If $(R^\bullet)_{\mathrm{red}}$ is dense in F, then R is weakly factorial if and only if $\widehat{R}$ is factorial. In this case,*

$$\mathcal{C}^*\big((R^\bullet)_{\mathrm{red}}, F\big) \cong \prod_{\mathfrak{p}\in P^*} \mathcal{C}^*\big(R_{\mathfrak{p}}^\bullet, \widehat{R_{\mathfrak{p}}^\bullet}\big) \cong \prod_{\mathfrak{p}\in P^*} (\mathcal{C}_1 \times \cdots \times C_{s_{\mathfrak{p}}})\,,$$

where for each $\mathfrak{p} \in P^*$, $\mathcal{C}_i = \Big\{[\mathfrak{P}_i(\mathfrak{p})]_{R_{\mathfrak{p}}^\bullet}^{\widehat{R_{\mathfrak{p}}^\bullet}}, [1]_{R_{\mathfrak{p}}^\bullet}^{\widehat{R_{\mathfrak{p}}^\bullet}}\Big\}$ for each $i \in [1, s_{\mathfrak{p}}]$ and $\big\{\mathfrak{P}_1(\mathfrak{p}), \ldots, \mathfrak{P}_{s_{\mathfrak{p}}}(\mathfrak{p})\big\}$ is the set of pairwise non-associated prime elements in $\widehat{R_{\mathfrak{p}}^\bullet}$.

Proof. For each $\mathfrak{p} \in \mathfrak{X}(R)$, it follows by Lemma 4.5 that $R_{\mathfrak{p}}^\bullet$ is a finitely primary monoid. Thus, all assertions follow by Proposition 4.3 and Theorem 4.4. $\square$

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